The Mathematics Educator
2001, Vol 6, No. 1, 76-85

Reading Mathematics

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Steen (1999) stated that 'quantitative literacy – or numeracy, as it is known in British English – means different things to different people'. He then proposed that quantitative literacy is both more than and different from mathematics – at least as mathematics has traditionally been viewed by school and society.

Yet quantitative literacy by any definition must include the abilities to read and write mathematics (Whitin and Whitin, 2000). Perhaps the debate should focus on the level of mathematics. Would an adult who can understand a chart published in the financial section of a newspaper but is unable to comprehend a written proof for the infinitude of primes be considered literate in the quantitative sense? There would be an unequivocal answer if the adult in question were a mathematics major. However, anecdotal knowledge suggests that undergraduate students generally do not and often cannot read mathematics textbooks and journals. When I once asked a writer of calculus textbooks for undergraduates if his students read his books, he replied, "At most only small portions and … the worked examples."

There is a growing awareness about the importance of reading and writing in mathematics (Kennedy, 1985; Bell and Bell, 1985; Withers, 1989; Turner, 1989). Waywood (1992) noted that the majority of reported work on writing to learn mathematics is focused at a primary level with the exceptions of Bell and Bell (1985) who studied the relation of writing to problem solving, and Borasi and Rose (1989) who reported on a college level algebra course where journals were kept. Similarly, work has been done on reading mathematics but again these are often in the primary school context and, as in Whitin and Wilde (1992), of the form where children's books are used to help both teachers and learners explore through the mathematical aspects of human experience and our physical world. Beery, Bressoud and McCray (2001) recommended in the CUPM Discussion Papers of the Mathematical Association of America that all students should be able to read mathematics and to communicate it both orally and in writing. This article discusses the need for teaching reading of mathematics at the university level. It also suggests some modes of assessment borrowed from the traditional teaching of language.

Mathematics as a Language and its Implications for Teaching

While some may make a distinction between language and mathematics (for example, Weinzweig (1982) called mathematics an "extension of language"), many would accept that mathematics is a language in the sense that it has its own notation, symbols and syntax (see for example, Usiskin (1996)).

My interest in the area of language in mathematics was stirred by an article by Leong (1995). Here, Leong made two observations about mathematics as a language which I shall quote. I shall then discuss the implications of these observations with regard to the teaching of reading.

Observation one: 'When I say that mathematics is a language, I do not mean the visualor even oral aspect of it. That is why it is still being written in English, Chinese, Japanese, Russian or whatever language you think in. The presence of a human linguistic element is really irrelevant. Just imagine a universal linguist (AUL for short) who is able to read any written human language on earth. Given a proof of a mathematical statement, would AUL be able to understand it? Would the mathematical statement itself make any sense to her? More importantly, would she be able to tell whether the proof is correct? If she could understand the proof, we would be inclined to think that she has been mathematically trained. If she could improve on the proof and rectify it, we would believe that she is a mathematician.' (p. 59)

Implication: At the university level, it should not matter what the fluency of the student is with regard to the language used to write the mathematics text. Neither should it matter if the text is gender biased or realistic. While not going so far as to say that these factors are totally inconsequential, I would suggest that by ignoring such peripheral issues, the teacher can concentrate on the mathematical comprehension of the passage vis-à-vis the reader's mathematical maturity and knowledge.

Observation two: 'We soon become aware that the language of mathematics has its own syntax (such as "If ..., then ...", There exists some ...", "Proof by contradiction") with a built-in thought process. In principle, each mathematical statement can be deduced from first principles ... However, because of the accumulative nature of the results, going back to first principles will be prohibitive in terms of time and space.' (p. 60)

Implication: Teaching students to read mathematics should involve teaching the syntax and thus the logic of the language. The student-reader also requires a metacognitive feature which constantly informs him of the body of results that he needs to understand a particular phrase or sentence.

To elaborate further on the second point above, a course on reading mathematics could include a section on the syntax of mathematics and some interesting mathematics topics which do not require much knowledge of prior results. Texts such as "How to read and do proofs" by Solow (1990) would be of signifant value for such a course.

Why teach reading?

Borasi and Rose (1989) wrote of the prevalent inappropriate approach to mathematics learning:

> One of the biggest challenges for mathematics instruction today is presented by the great number of students showing an inappropriate approach to the subject and its learning. Most mathematics students seem to interpret their role as essentially acquiring (i.e., memorizing) facts and algorithms that can be immediately applied to the solution of given exercises; few students expect mathematics to be meaningful and fewer still see mathematics as a creative undertaking. Consequently, students are too often content with externally manipulating symbols and doing routine problems, without ever reaching a deep and personal understanding of the material. Unfortunately, even though these attitudes and expectations may allow some students partial short-term successes, they are not conducive to the development of conceptual understanding and problem solving skills necessary to succeed in mathematics in the long run. (p. 347)

They suggested that the use of *writing to learn* can provide a valuable means to facilitate a personalized and making-of-meaning approach to learning mathematics. Similarly, one may suggest *reading to learn* as another component to learning mathematics. Reading is a precursor to writing. Thus, if 'suggesting an intensive use of writing in mathematics courses might seem at first surprising [since] few things stand so far apart in students' minds as mathematics and writing, and traditionally, the amount of writing required in mathematics courses has been minimal' (Borasi and Rose, 1989), then a course in reading, it being naturally 'somewhere in between' mathematics and writing, would be a less surprising suggestion. Borasi and Rose advocate a journal form of writing mathematics that is more like writing *about* mathematics. Here however, I propose reading mathematics and not reading *about* mathematics.

That reading has been sidelined with regard to learning mathematics is often a shocking realisation to even mathematics educators. Many teachers have learned to live with the fact that most of their students do not read mathematics text but rely solely on sketchy notes taken during lectures and worked examples (and hints) to prepare for exams. It is typical to advocate for the general population a level of quantitative literacy where one is able to read figures, charts and graphs in everything from newspapers to medical reports. While this level may be suitable for the general population, this falls far short of being able to read mathematics and is too low a target for mathematics majors. That many mathematics graduates today minimize contact with mathematics may be a direct result of the low reading expectations required of them when they were undergraduates. Typically, the best students do not need formal instruction with regard to reading but the number of such students is small. Most students require specific instruction and guidance to overcome the many obstacles to comprehension of mathematics text. There is thus a need for some planned instruction and formative evaluation on reading.

It is good that the emphasis on quantitative literacy should be the uplifting of the math-phobic and the general reading public with its emphasis in the primary and secondary schools. However, it would be shortsighted to neglect the uplifting of those with the potential to do higher level mathematics or even to appreciate mathematics as a form of art. Although not all with such a potential will turn out to be professional mathematicians, teaching students to read mathematics will create a critical core of math-literates who will form a lively community that can appreciate mathematics. This community may turn out to be crucial to the promotion of mathematics by funding and encouraging promising students and mathematicians or by being a market for mathematics books and seminars. Classical music endures for a similar reason that music education creates a community of music lovers. But unlike music, most mathematics has to be read.

Methods of assessment

A key factor in improving the reading of mathematics is appropriate and adequate assessment. A course on reading *per se* like a course on problem solving *per se* would have only a short-term effect. Assessment of reading must be conducted in as many courses as possible. Mathematics courses should involve reading assignments and therefore lecturers must alter assessment practices to include the assessment of reading.

However, I suggest that there is no need to reinvent the wheel with regard to assessment of achievement in reading mathematics. Specifically, methods of assessment that are already in use in the teaching of language could be used. Consider the following examples.

1. The Cloze passage

One method of assessment is the Cloze passage, i.e., a passage with parts removed and replaced with blanks which are to be filled in by the student. Below is an example of a Cloze passage. This was used in a third year undergraduate number theory examination paper.

Consider the following theorem and its proof. In your answer sheet, write down the missing parts indicated by the underlined bracketed roman numerals.

Theorem: If a and b are not both zero and if $d = (a, b)$, then d is the least element in the set of all positive integers of the form $ax+by$.

Proof : Consider the set C of all positive integers of the form $ax + by$. By hypothesis, at least one of a and b is different from zero. For definiteness, suppose that $a \neq 0$. If $a > 0$, then a itself is a member of C, and if $a < 0$, then __(i)__ is a member of C. Therefore, C is not empty, and so, by the __(ii)__ principle, must have a least element. Let $e = ax_0 + by_0$ be the least element of C. It suffices to show that $d = e$.

By __(iii)__, there exist integers q and r with $0 \leq r < e$ such that $a = eq + r$. Thus, $r = a - eq = a - (ax_0 + by_0)q = a(1 - qx_0) + b(-qy_0)$, which is of the form $ax + by$. If $r \neq 0$, then it would be a member of __(iv)__, and since $r < e$, this would contradict our assumption that __(v)__. Thus, $r = 0$ and so $a =$ __(vi)__. This implies that $e \mid a$. Similarly, one can show that $e \mid$ __(vii)__. Thus, e is a __(viii)__ of a and b, so that by the definition of the greatest common divisor, we have e __(ix)__ d. On the other hand, since $e = ax_0 + by_0$ and $d \mid a$ and $d \mid b$, it follows that __(x)__ and so $d \leq e$. Thus, finally we have $d = e$, as was required.

The knowledge required to complete the Cloze passage above are the statements of the Well-Ordering Principle and the Division Algorithm, and the definition of the greatest common divisor. One student taking the paper gave the following answers:

b ; mathematical induction ; division algorithm ; C ; $r \in C$; eq ; (a, b) ; common divisor ; $<$; $d \mid e$.

The answers of the students are good indicators of reading ability and comprehension. Half of the answers are correct (readers are invited to attempt the passage themselves). In this case, the wrong answers show as well that the student understands the syntax of the language. This can be seen from the 'ease of reading' even when the wrong answers are inserted. An analogy with regard to the English language is given below. Consider the following sentences:

Singapore is a green <u>alien</u>.
Singapore is a green <u>terribly</u>.

Although both sentences are semantically wrong, the first sentence is syntactically correct. Most of the missing words in the Cloze passage are nouns (seven out of ten). It would be a better assessment of students' reading ability if different parts of a mathematical sentence such as conjunctions, verbs, adjectives, phrases and clauses are tested more. This was done in the next Cloze example.

Although the passage above was given as part of a summative evaluation, the Cloze passage can be used for formative evaluation and as exercises in the teaching of reading mathematics or mathematics itself. Below is another Cloze passage that was recently given to students taking a course on Problem Solving and Discrete Mathematics in a Masters of Education (Mathematics Education) program:

Theorem: Let G be a nontrivial connected graph. Then G has the unicursal property if and only if every vertex of G is even.

Proof : Let G be a graph with the unicursal property. Thus, G has a ___________ C. Let v be an arbitrary vertex of G. If v is not the initial (and thus not the final) vertex of C, then each time v is encountered on C, it is entered and left by means of __________ vertices. Thus, each occurrence of v in C represents a contribution of ______ to the degree of v so that v has ______ degree. If v is the initial vertex of C, then C, then C begins and ends with v, each term representing a contribution of _____ to its degree while every other occurrence of V indicates an addition of ________ to its degree. This gives an _____________ to v.

_____________, let G be a nontrivial connected graph in which every vertex is even. We employ induction on the number q of edges of G. For $q = 3$, the smallest possible value, there is only one such graph; namely _____, and this graph has the unicursal property. Assume that all nontrivial connected graphs having only even vertices and with fewer than q edges, $q >$ ____, have the unicursal property; and let G be such a graph with q edges.

Select some vertex u in G, and let W be a u-u circuit of G. Such a circuit exists in G since if W' is any u-v trail of G with $u \neq v$, then necessarily, an ______ number of edges of G incident with ______ are present in W', implying that W' can be extended to a trail W'' containing ______ edges than that of W'. Hence W' can be extended to ____________________.

If the circuit W contains every edge of G, then ____________. Otherwise, there are edges of G not in _____. Remove from G all those edges which are in W together with any resulting isolated __________, obtaining the graph G'. Since each vertex of W is incident with an ______ number of edges of _____, every vertex of G' is ______. Every component of G' is a nontrivial graph with fewer than ______ edges and __________ by hypothesis. Since G is ________, every component of G' has a vertex that also belongs to W. Hence a circuit of G that contains all the edges of G can be constructed by ______________________________________.

The knowledge required to complete the Cloze passage above are basic graph theory and the then just introduced notion of the unicursal property of a multigraph, i.e., the possession of a closed walk which passes through each edge in the multigraph exactly once.

The students were each given the passage and, as the instructor, I went through the proof in a round-table discussion manner. As the reader will notice, the answers here are sometimes quite long. The students agreed that this format of presenting a proof forced them to think much deeper and also helped them in reading mathematics.

2. Comprehension Passage

Another method of assessment borrowed from language learning is the comprehension passage. Typically, a totally new passage is presented and questions are asked on the passage. In evaluating the ability to read mathematics, a proof of a theorem that is unfamiliar to the student may be presented and questions will have to be answered. Of course, all prerequisite knowledge, notations and terminology must have been covered in the course. As it stands, many mathematics tests are based on theorems and formulas taught in class and generally not totally new ones. Hence, it is not always possible to test the ability to read mathematics.

If we were to present a proof such as the second passage above (theorem on the unicursal property), some possible test items may be as follows:
Does the first paragraph of the proof show sufficiency or necessity?
Why is C_3 the only nontrivial connected graph with 3 edges in which every vertex is even?

Distinguish, based on the passage, between a circuit and a trail.
What is the induction hypothesis used?

Based on the last paragraph, construct a closed walk which passes through each edge in the given graph (diagram not shown in this article) exactly once.

Note that the questions above are mostly not of recall level. Indeed, they require an understanding of proof techniques (as in the first and fourth items), understanding of definitions and concepts (as in the second item), and the ability to apply the consequences of the proof (as in the last item).

To give an added dimension to evaluating comprehension, one may purposely include errors in the passage and require the student to spot and rectify them. One may also ask for corollaries to the given theorem.

3. Analogies

Rubenstein (1996) provided additional strategies to support the learning of the language of mathematics. Some, which she had found successful, included inviting students to invent their own terms, exploiting analogies and metaphors, using charts, and revealing the origins of language. The strategy, which I think could be adapted as an assessment option for university mathematics, is that of using analogies. Completing exercises like the following (an example for graph theory) at appropriate points in students' studies builds connections between known and new ideas and invites higher-level thinking.

circuit : Eulerian :: cycle : ________
vertex : edge :: chromatic number : ________
vertex : edge :: ________ : bridge
2-colourable : 4-colourable :: even cycle : ________
graph : degree :: digraph : ___________

(Solutions: Hamiltonian, chromatic index, cut-vertex, planar graph, in-degree and out-degree.)

Conclusion

There is a quantum leap between the mathematical reading ability and interests of mathematics teachers at the school level and those at the university and research level. This great difference is reflected in the degree of difficulty of mathematics at these levels. I propose that to smooth the transition between the two levels, university mathematics departments should include components on reading mathematics into first year university courses followed by appropriate evaluation of reading ability and comprehension in exams and tutorials.

This paper represents my preliminary research on reading mathematics. As a follow-up to this paper, I intend to obtain some quantitative data on the reading ability of mathematics majors and the effects, if any, that a reading course would have on proficiency.

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